A Legendre Approximation Method for the Circular Microstrip Disk Problem

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Abstract—**The quasi-static solution for the circular microstrip disk is studied using a Galerkin solution to the Fredholm integral equation of the first kind derived by using the Green's function approach. The basis functions are modified Legendre polynomials combined with a reciprocal square root to provide the correct singularity in charge density at the edge of the disk. The integrals involving the singular part of the Green's function are evaluated exactly, the remainder by using Gaussian quadrature. The method is compared in computational efficiency with recent methods based either on a Galerkin approach in the spectral domain, or the use of dual integral equations. Numerical results are given for charge distribution and capacitance; they are compared to exact results and those obtained by others, and the limitations of those methods are discussed. Closed form expressions are given for the capacitance of a disk based on two simple charge distributions.**

I. Introduction

Recently, three methods have been proposed for the determination of the capacitance of the circular microstrip disk for applications in microwave integrated circuits. Itoh and Mittra [1] propose a Galerkin technique in the spectral domain. They present a general theory but give numerical results only with first-order approximation in their technique. The coefficients of the Galerkin matrix in [1] involve infinite integrals of Bessel functions, and these require considerable computer time. To avoid this deficiency, they present numerical results only for two very simple one-term approximations to the charge density, but have not shown what the contribution is of additional basis functions to capacitance results. We suggest that although working in the spectral domain makes the formulation of electrostatic problems easy, it has no advantage over the space formulation, used here in the numerical solution. The more accurate calculations of the present analysis confirm, however, that the one-term results for capacitance given in [1] are sometimes remarkably accurate. In addition, it is shown in the Appendix that the capacitances corresponding to these one-term approximations may be expressed in closed forms which exhibit the limiting properties observed numerically in [1] and take considerably less computation time.

Borkar and Yang [2] present a method based on dual integral equations. Although they claim that their numerical results are "slightly lower" than those of [1], the disparity is more than 10 percent in some cases. Numerically, their method is less efficient than that of [1] since it involves the calculation of numerous infinite integrals of products of Bessel functions. Their claim that the solution is obtained from a "quickly convergent series" is not substantiated; on the basis of our experience with series of the kind they use in other, but similar applications, we doubt the validity of this claim.

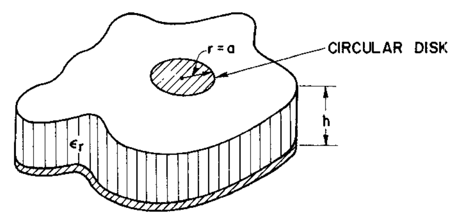

Fig. 1. The geometry of the circular microstrip disk.

Although the nature of the charge distribution is incorporated in [1] or [2], the calculation of charge distribution would require a number of basis functions, much computational effort, and time. From a practical point of view, this is not a deficiency since in practice the exact charge distribution is rarely required but solutions to electrostatic problems are incomplete without it.

The capacitance of the circular microstrip disk *in vacuo* for large a/h may be obtained approximately by a formula due to Kirchhoff [12]. Wolff and Knoppik [13] assume that for large a/h the edge field of the circular microstrip disk is similar to that of the rectangular microstrip disk. They use Kirchhoff's [12] formulas, and further assume that the edge effect may be computed from the *quasi-TEM* solution of a microstrip line of width $w = 2a$. In this way they obtain a first-order modification of Kirchhoff's formula valid for a certain range of a/h.

In this paper the electrostatics of the circular microstrip disk is formulated as a Fredholm integral equation of the first kind with a singular kernel. The charge density is expanded in terms of modified Legendre polynomials, together with a reciprocal square root to provide the correct singularity at the edge of the disk. The integrals arising from the singular part of the kernel are evaluated exactly, the remainder by using Gaussian quadrature. The total charge and capacitance are obtained with no extra numerical integrations.

II. Formulation of the Basic Integral Equation

The axially symmetric electrostatic potential ψ due to a charged circular disk shown in Fig. 1 satisfies Laplace's equation and certain boundary conditions. The Green's

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function solution for $\psi(r)$ in terms of charge distribution $\sigma(s)$ is most easily obtained by transforming to the spectral domain. The Hankel transform of the potential $\hat{\psi}(\xi)$ may be expressed in terms of the transform $\hat{\sigma}(\xi)$ in the form given by Itoh and Mittra [1], namely,

$$\hat{\psi}(\xi) = g(\xi)\hat{\sigma}(\xi) \tag{1}$$

where, in the particular problem shown in Fig. 1,

$$g(\xi) = (1 - e^{-2\xi h})/[\alpha\xi(1 - \beta e^{-2\xi h})] \tag{2}$$

$\alpha = \varepsilon_0(1 + \varepsilon_r)$ and $\beta = (1 - \varepsilon_r)/(1 + \varepsilon_r)$. Now write

$$\hat{\sigma}(\xi) = \int_0^a s\sigma(s)J_0(s\xi)\, ds \tag{3}$$

and apply the inverse Hankel transform to both sides of (1); the result is the following Fredholm equation of the first kind:

$$\alpha\psi(r) = \int_0^a sG(r,s)\sigma(s)\, ds, \qquad r \le a. \tag{4}$$

Here $\psi(r)$ is the potential on the disk, $\sigma(s)$ is the unknown charge distribution, and $G(r,s)$ is the axisymmetric Green's function linking the potential at radius r to an annual distribution $\sigma(s)$ at radius s. $G(r,s)$ may be written

$$G(r,s) = k_0(r,s) + k_1(r,s) \tag{5}$$

where $k_0(r,s)$ is the axisymmetric free-space Green's function

$$k_0(r,s) = \int_0^\infty J_0(r\xi)J_0(s\xi)\, d\xi = \begin{cases}(2/\pi s)K(r/s), & r < s\\ (2/\pi r)K(s/r), & r > s\end{cases} \tag{6}$$

$$k_1(r,s) = -(1 - \beta)\int_0^\infty \frac{e^{-2\xi h}}{1 - \beta e^{-2\xi h}} J_0(r\xi)J_0(s\xi)\, d\xi \tag{7}$$

in which $K(\cdot)$ is the complete elliptic integral of the first kind. If $\psi(r) = 1$ for $r \le a$, then the capacitance is the total charge given by

$$C = 2\pi\int_0^a r\sigma(r)\, dr. \tag{8}$$

The equation governing the charge distribution on the disk in any other axisymmetrical configuration, e.g., when there is a cover over the disk, will have the same form (4) where $k_0(r,s)$ will be the same but the multiplier of $J_0(r\xi)\cdot J_0(s\xi)$ in the integral for $k_1(r,s)$ will be changed. The analysis given as follows may be modified to apply to such problems.

The term $(1 - \beta e^{-2\xi h})^{-1}$ in (7) may be expanded into a uniformly convergent series, and then (7) may be expressed as

$$k_1(r,s) = -(1 - \beta)\sum_{n=1}^{\infty} \beta^{n-1}\int_0^\infty e^{-2\xi nh}J_0(r\xi)J_0(s\xi)\, d\xi \tag{9}$$

which in turn may be written in terms of elementary functions using the result given by Watson [3]; the final result is

$$k_1(r,s) = \frac{-2}{\pi}(1 - \beta)\sum_{n=1}^{\infty} \beta^{n-1}f_n^{-1}K\left\{\frac{2\sqrt{rs}}{f_n}\right\} \tag{10}$$

where $f_n = [(r + s)^2 + (2nh)^2]^{1/2}$. The elliptic integrals are computed using the Chebyshev expansion given by Luke [4], and the infinite series is summed by applying the nonlinear transformations developed by Levin [5]. These were found to be fast and accurate.

III. Approximation Solution of the Integral Equation Using Properties of Legendre Polynomials

Introduce dimensionless variables and functions by taking

$$r = ax \qquad s = ay \qquad \sigma(s) = V_0\alpha a^{-1}\eta(y)$$
$$\psi(r) = V_0 g(x) \qquad k_0(r,s) = a^{-1}k_0{}^*(x,y)$$
$$k_1(r,s) = a^{-1}k_1{}^*(x,y) \tag{11}$$

where V_0 is some standard potential. Then (4) and (5) give

$$g(x) = \int_0^1 [k_0{}^*(x,y) + k_1{}^*(x,y)]y\eta(y)\, dy, \qquad 0 \le x \le 1. \tag{12}$$

Now assume that $\eta(y)$ may be approximated by

$$\eta(y) = (1 - y^2)^{-1/2}\sum_{n=0}^{N} a_n P_n{}^*(y) \tag{13}$$

where $P_n{}^*(y) \equiv P_{2n}(\sqrt{1 - y^2})$, and $P_{2n}(\cdot)$ is the Legendre polynomial of degree $2n$, and a_n are coefficients to be determined. Substitute (13) into (12): the equation becomes

$$g(x) = \sum_{n=0}^{N} a_n \int_0^1 [k_0{}^*(x,y) + k_1{}^*(x,y)]yP_n{}^*(y)(1 - y^2)^{-1/2}\, dy. \tag{14}$$

The first integral, involving the singular part of the kernel, may be evaluated in closed form by using the result due to Popov [6], namely,

$$\int_0^1 (1 - y^2)^{-1/2}yk_0{}^*(x,y)P_n{}^*(y)\, dy = \lambda_n P_n{}^*(x) \tag{15}$$

where

$$\lambda_n = (\pi/2)[(2n)!/\{2^{2n}(n!)^2\}]^2. \tag{16}$$

The second integral in (14) may be transformed by the substitution $t = \sqrt{1 - y^2}$ into

$$\int_0^1 k_1{}^*(x,y)yP_n{}^*(y)(1 - y^2)^{-1/2}\, dy = \int_0^1 k_2(x,t)P_{2n}(t)\, dt = b_n(x) \tag{17}$$

where $k_2(x,t) = k_1{}^*(x,y)$. This integral may be approximated using ordinary Gaussian quadrature of suitable order $2M > 2N$. If w_j,t_j are the weights and positive abscissa of this quadrature given, for example, in Abramowitz and Stegun [7], then

$$b_n(x) \simeq \sum_{j=1}^{M} w_j k_2(x,t_j) P_{2n}(t_j). \tag{18}$$

Now (15) and (17) reduce (14) to a set of linear equations

$$g(x) = \sum_{n=0}^{N} a_n \lambda_n P_n^*(x) + \sum_{n=0}^{N} a_n b_n(x), \qquad 0 \le x \le 1. \tag{19}$$

These equations may be solved by multiplying throughout by $x(1-x^2)^{-1/2}P_m^*(x)$ and using the orthogonality conditions

$$\int_0^1 x(1-x^2)^{-1/2} P_n^*(x) P_m^*(x)\,dx = \int_0^1 P_{2n}(u) P_{2m}(u)\,du = (4m+1)^{-1}\delta_{m,n} \tag{20}$$

and another Gaussian quadrature

$$\int_0^1 x(1-x^2)^{-1/2} P_m^*(x) b_n(x)\,dx = \int_0^1 P_{2m}(u) d_n(u)\,du = c_{nm} \tag{21}$$

where $d_n(u) \equiv b_n(\sqrt{1-u^2})$ and

$$c_{nm} \simeq \sum_{k=1}^{M} w_k P_{2m}(t_k) d_n(t_k). \tag{22}$$

Thus (19) yields

$$(4m+1)^{-1}\lambda_m a_m + \sum_{n=0}^{N} c_{nm} a_n = \delta_{m,0}, \qquad m = 0,1,\cdots,N. \tag{23}$$

The right-hand side corresponds to a dimensionless potential $g(x) = 1$. If the same order of Gaussian quadrature is used in (18) and (22), then c_{nm} will be symmetric.

Equation (23) provides $N+1$ equations for the determination of the $N+1$ unknown a_n. The capacitance is given by

$$C = 2\pi a \alpha a_0. \tag{24}$$

Note that no extra numerical integration is required to compute this quantity.

IV. Variational Properties of the Capacitance

In this section it will be shown that the capacitance given by (24), where a_0 is obtained from the solution of (23), is stationary for arbitrary variations in $\eta(y)$ as given by (13), that is, in a_n.

First in (12) put $g(x) = 1$, $0 \le x \le 1$, and multiply both sides of this equation by $\int_0^1 x\eta(x)\,dx$; this gives

$$\int_0^1 x\eta(x)\,dx = \int_0^1 \int_0^1 K^*(x,y)\eta(x)\eta(y) yx\,dy\,dx \tag{25}$$

where $K^* = k_0^* + k_1^*$. Here the potential on the disk is 1 and the total charge Q is from (8) and (11) given by

$$Q = 2\pi\alpha a \int_0^1 y\eta(y)\,dy. \tag{26}$$

If we write $Q = C$, then (25) shows that

$$\frac{2\pi\alpha a}{C} = \frac{\int_0^1\int_0^1 K^*(x,y)\eta(x)\eta(y)xy\,dy\,dx}{[\int_0^1 x\eta(x)\,dx]^2}. \tag{27}$$

Since $K^*(x,y)$ is symmetric, then the right-hand side of (27) is always positive. To make the left-hand side of (27) stationary with respect to arbitrary variations in the functional form of η, substitute the expansion (13) for η and set $\partial C/\partial a_n \equiv 0$; the result is

$$\frac{2\pi\alpha a}{C} a_0 \delta_{n,0} = (4n+1)^{-1}\lambda_n a_n + \sum_{j=0}^{N} c_{jn} a_j, \qquad n = 0,1,2,\cdots,N \tag{28}$$

where use was made of (15) and (21). But, in view of (24), the capacitance is given by

$$C = 2\pi\alpha a a_0 \tag{29}$$

from which the left-hand side of (28) becomes $\delta_{n,0}$. Now comparing (28) and (23), we see that they are identical. So the capacitance as given by (24) is stationary with respect to arbitrary variations in the functional form of η, i.e., $\partial C/\partial a_n = 0$ for $n = 0,1,2,\cdots,N$.

Further, since the right-hand side of (27) is always positive and the left-hand side is stationary, then the exact solution for capacitance, say C_∞, will be always greater than or equal to C_N; that is, $C_\infty \ge C_N$ and thus the approximate capacitance from the present analysis C_N, where N denotes the number of terms in the expansion (13), is a lower bound.

V. Formulas for Capacitance

For the circular microstrip disk problem, Itoh and Mittra [1] obtained results with first-order approximation in their technique, based on two very simple one-term approximations to the charge distribution. In the Appendix, these are expressed in closed form. The result, for the capacitance associated with the Maxwell function approximation to the charge density, is from (A7) given by

$$C_M = 2\pi a\alpha \Big/ \left[\pi/2 - (1-\beta)\sum_{n=1}^{\infty} \beta^{n-1} I_n\right] \tag{30}$$

where

$$I_n = \frac{1}{2}\left[\pi - 2\tan^{-1}(n\gamma) + n\gamma \ln\left\{\frac{n^2\gamma^2}{1+n^2\gamma^2}\right\}\right]$$

and $\gamma = h/a$. The corresponding result for the gate function approximation to the charge density is from (A11) given by

$$C_G = \frac{\pi\alpha a}{2} \Big/ \left\{4/3\pi - (1-\beta)\sum_{n=1}^{\infty} \beta^{n-1} J_n\right\} \tag{31}$$

where

$$J_n = [4/(3\pi k^3)]\{(2k^2-1)E(k) + (1-k^2)K(k)\} - n\gamma$$

$$k^2 = 1/(1+n^2\gamma^2).$$

The computation of capacitance via the formulas (30) and (31) takes considerably less computation time than the infinite integrals in [1]; a comparison of the results for

TABLE I
NORMALIZED CAPACITANCE FOR THE CIRCULAR MICROSTRIP *in Vacuo*

h/a	N = 0	N = 1	N = 2	Nomura – Cooke
0.2	1.5290	1.5800	1.5800	1.5802
0.5	2.3094	2.3183	2.3183	2.3183
1.0	3.5336	3.5345	3.5345	3.5312
2.50	7.2684	7.2684	7.2684	7.2683
5.00	13.5920	13.5920	13.5920	13.5918
10.00	26.3006	26.3006	26.3006	26.2771

capacitance obtained by (30) and (31) and by [1] shows that the numerical integration used in [1] led to a capacitance error somewhat less than 1 percent.

VI. EXAMPLES

The simplest problem is the isolated disk, obtained by putting $h \to \infty$ in Fig. 1, for which the exact solution is known. Here, $k_1(r,s) \equiv 0$ and (14) shows that

$$\sum_{n=0}^{N} a_n \int_0^1 k_0^*(x,y)P_n^*(y)(1 - y^2)^{-1/2}y\,dy = g(x), \qquad 0 \le x \le 1 \quad (32)$$

which in view of (23) with $c_{nm} \equiv 0$ gives $a_0 = \lambda_0^{-1} = (2/\pi)$ and $a_n \equiv 0$ for $n \ge 1$. The charge distribution in this case is from (13) and (11) given by

$$\sigma(r) = \begin{cases} 2\pi^{-1}V_0\varepsilon_0(1 + \varepsilon_r)(a^2 - r^2)^{-1/2}, & \text{for } r \le a \\ 0, & \text{for } r > a \end{cases} \quad (33)$$

whereas the capacitance is simply

$$C = 4\varepsilon_0(1 + \varepsilon_r)a. \quad (34)$$

The present analysis therefore gives the exact solution to the isolated disk problem.

Another simple problem is the microstrip disk *in vacuo*, obtained by putting $\varepsilon_r = 1$ in Fig. 1, for which no exact solution is known to the authors. Numura [8] obtained an approximate solution to Love's integral equation and gave a table of values proportional to the capacitance of the disk for $h/a \ge 0.2$; some of these were later corrected by Cooke [9]. In this case the only nonzero term in $k_1(r,s)$ is, from (10), given by

$$k_1(r,s) = -\frac{2}{\pi}f_1^{-1}K\left\{\frac{2\sqrt{rs}}{f_1}\right\}. \quad (35)$$

Table I shows values of the normalized capacitance $C^* = hC/\pi a^2\varepsilon_0$ obtained from the present analysis, with $k_1(r,s)$ as given by (35); N denotes the order of the modified Legendre polynomials used in the expansion (13) for the unknown charge distribution. It will be noted that results converge rapidly and that Numura–Cooke values are in agreement with those obtained from the present analysis. Table I also verifies the stationary properties of the capacitance derived in Section V. These have been two test cases to show the accuracy and convergence of the capacitances obtained from the present analysis.

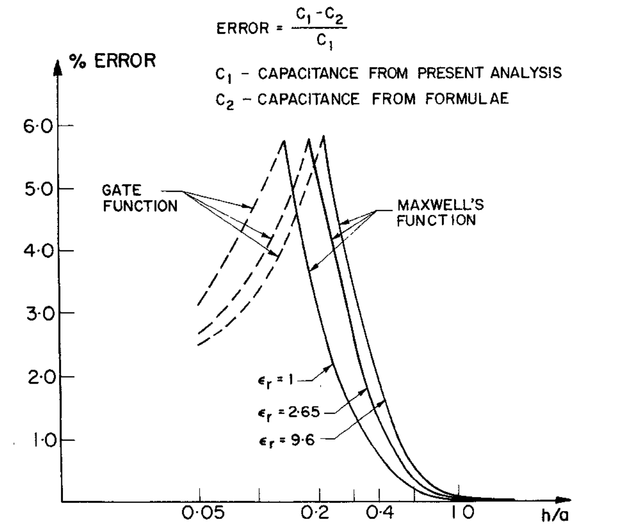

Fig. 2. The percentage error in the capacitance obtained from the formulas [e.g., (30) and (31)].

VII. THE ACCURACY OF FORMULAS AND RESULTS FOR THE CAPACITANCE AND RESONANT FREQUENCY

In order to verify the accuracy of the formulas associated with the Maxwell function approximation (30) or the gate function approximation (31), a comparison is made with the more accurate results of the present analysis. The percentage error in the capacitance obtained from the formulas is plotted in Fig. 2 for three different dielectric constants and for $h/a \ge 0.05$. Note that the maximum error occurs in the neighborhood of $h/a = 0.2$ where neither the gate nor the Maxwell function may adequately approximate the actual charge distribution; both lead to ~ 5.8 percent in capacitance error. Fig. 3 shows three dimensionless charge distributions; for $h/a = 0.5$, which is large enough for the Maxwell function to be accurate; for $h/a = 0.22$, for which neither function is accurate; and for $h/a = 0.1$ for which the gate function is adequate.

The results for capacitance given by Borkar and Yang [2] are always lower than those obtained by [1], and for some h/a the disparity exceeds 10 percent. The results obtained by [1] are lower than those obtained by the present analysis, and for h/a in the neighborhood of 0.2 they are lower by ~ 5.8 percent. The more accurate capacitances obtained from the present analysis are therefore shown in Fig. 4, together with the resonant frequency of the dominant TM_{110} mode, based on a circular disk resonator with magnetic side walls. This resonant frequency f_{110} is linked to the normalized capacitance $C^* = hC/\pi a^2\varepsilon_0\varepsilon_r$ via the relation [10].

$$f_{110} = \frac{vk}{2\pi a\sqrt{\varepsilon_r C^*}} \quad (36)$$

where $k = 1.841$ and v is the speed of light *in vacuo*. The experimental results for the resonant frequency f_{110}

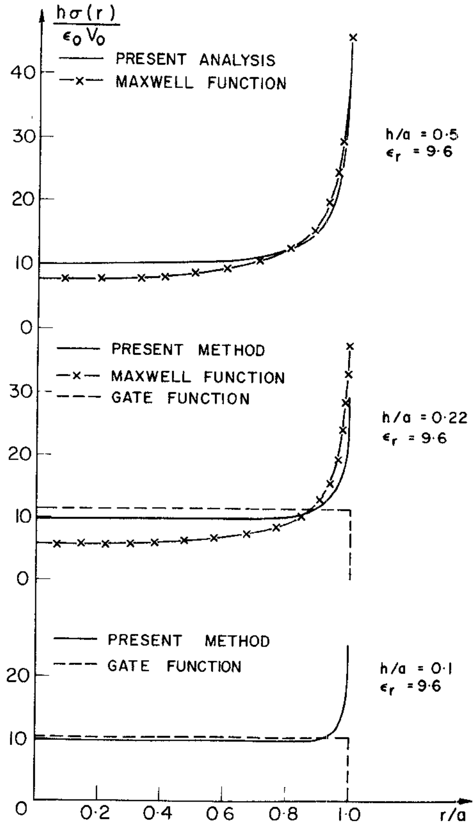

Fig. 3. The Maxwell and gate function approximation versus the actual charge distribution.

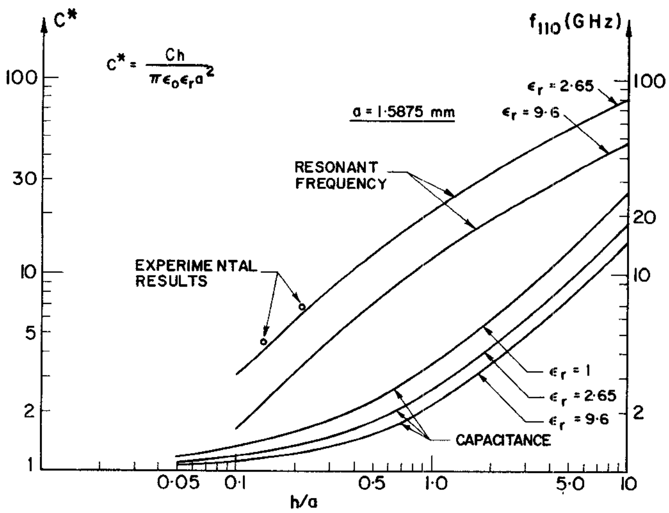

Fig. 4. The normalized capacitance and the resonant frequency of the dominant TM_{110} of the disk resonator. The results have been computed with $N \leq 5$ in the expansion (13) for the charge distribution.

reported in [10] for $\varepsilon_r = 2.65$ are also included in Fig. 4 for comparison. We note, however, that these experimental results are in the range of h/a from 0.14 to 0.22, and Fig. 2 shows that this is the worst range for Itoh and Mittra's trial functions, although they obtained almost better results with respect to the experimental data. It will also be noted that although Borkar and Yang [2] have obtained very good agreement with the experimental results reported in [10], their capacitances are not accurate as they stated and as verified by comparing with the results from the present analysis.

VIII. Conclusion

A rigorous and yet simple, fast, and accurate solution of the electrostatic problem of the circular microstrip disk has been presented. Closed form expressions have been derived for the capacitance of the disk based on two simple charge distributions.

The capacitance may be computed in three different ways: either by using the present analysis, or by using the formulas derived in the Appendix together with the correction curves presented in Fig. 2, or, alternatively, by using the Galerkin's procedure suggested by Itoh and Mittra and then the correction curves of Fig. 2.

In either case, the results may be used to obtain approximately the resonant frequency of the dominant TM_{110} mode of the circular disk resonator. When a full wave analysis is available, the results for resonant frequency should be compared with those obtained from the present analysis, in order to verify whether the quasi-static model of the disk resonator may adequately approximate the dynamic model.

Appendix

In [1] Itoh and Mittra consider two possible one-term charge distributions for the disk of radius a:

a) Maxwell function

$$\sigma(r) = \frac{Q}{2\pi a}(a^2 - r^2)^{-1/2} \tag{A1}$$

b) Gate function

$$\sigma(r) = \frac{Q}{\pi a^2}. \tag{A2}$$

Their results for the capacitance may be obtained by inserting either expression a) or b) for $\sigma(s)$ in (4), putting $\psi(r) = 1$, multiplying both sides of the equation by $r\sigma(r)$, and integrating over $(0,a)$. Their result for the Maxwell's function is the same as that obtained in this paper for one term in the Legendre expansion (13) when the integrations are carried out exactly; it is

$$\alpha a = \frac{Q}{2\pi a}\int_0^a r(a^2 - r^2)^{-1/2}\,dr \int_0^a s(a^2 - s^2)^{-1/2} G(r,s)\,ds. \tag{A3}$$

The expansion for $G(r,s)$ is

$$G(r,s) = \int_0^\infty J_0(r\xi)J_0(s\xi)\left\{1 - (1-\beta)\sum_{n=1}^{\infty}\beta^{n-1}e^{-2\xi nh}\right\}d\xi. \tag{A4}$$

Thus (A3) may be rewritten

$$\alpha a = \frac{Q}{2\pi a}\int_0^\infty \left[\frac{\sin(a\xi)}{\xi}\right]^2 \left\{1 - (1-\beta)\sum_{n=1}^{\infty}\beta^{n-1}e^{-2\xi nh}\right\}d\xi. \tag{A5}$$

Integrating the Laplace transform 29.3.110 of [7], we find

$$I_n = \int_0^\infty \left(\frac{\sin x}{x}\right)^2 e^{-2n\gamma x}\, dx$$
$$= \frac{1}{2}\left[\pi - 2\tan^{-1}(n\gamma) + n\gamma \ln\left\{\frac{n^2\gamma^2}{1+n^2\gamma^2}\right\}\right] \quad \text{(A6)}$$

so that with $\gamma = h/a$ we find

$$C = 2\pi a\alpha \Big/ \left[\pi/2 - (1-\beta)\sum_{n=1}^{\infty} \beta^{n-1} I_n\right]. \quad \text{(A7)}$$

This expression exhibits the correct limiting behavior of the capacitance for large h/a, but not for small h/a. Thus as $h/a \to \infty$, $C \to 4a\alpha$.

The corresponding result for the gate function is

$$\frac{\alpha a^2}{2} = \frac{Q}{\pi}\int_0^\infty \left(\frac{J_1(\xi a)}{\xi}\right)^2 \left\{1 - (1-\beta)\sum_{n=1}^{\infty}\beta^{n-1}e^{-2\xi nh}\right\} d\xi. \quad \text{(A8)}$$

Now I_n is replaced by J_n, where

$$J_n = \int_0^\infty \left(\frac{J_1(x)}{x}\right)^2 e^{-2n\gamma x}\, dx \quad \text{(A9)}$$

and Watson [3] gives

$$J_n = \frac{1}{\pi}\int_0^\pi \{\sqrt{n^2\gamma^2 + \sin^2(\phi/2)} - n\gamma\}(1 + \cos\phi)\, d\phi. \quad \text{(A10)}$$

This may be evaluated by using the transformation given by Byrd and Friedman [11, §282]; the result is

$$J_n = [4/(3\pi k^3)]\{(2k^2 - 1)E(k) + (1 - k^2)K(k)\} - n\gamma$$

$k^2 = 1/(1 + n^2\gamma^2)$, $J_0 = 4/3\pi$, and $E(\cdot)$ is the complete elliptic integral of the second kind. Thus

$$C = \frac{\pi\alpha a}{2} \Big/ \left\{(4/3\pi) - (1-\beta)\sum_{n=1}^{\infty}\beta^{n-1}J_n\right\}. \quad \text{(A11)}$$

This expression exhibits the correct limiting behavior of the capacitance for small h/a, but not for large h/a. Thus when h/a is small,

$$J_n = 4/(3\pi) - n\gamma + 0(\gamma^2) \quad \text{(A12)}$$

so that, for small h/a,

$$C = \frac{\pi\alpha a}{2}\Big/\left\{\frac{(1-\beta)\gamma}{(1-\beta)^2}\right\} = \frac{\pi\alpha a(1-\beta)}{2\gamma}. \quad \text{(A13)}$$

In view of $\alpha(1 - \beta) = 2\varepsilon_0\varepsilon_r$, C has the known value of the parallel plate condenser, namely,

$$C = \pi a^2 \varepsilon_0 \varepsilon_r / h. \quad \text{(A14)}$$

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